Practical Tools for OR, ML and DS

Lecture $9\frac{1}{2}$: Inventory #3 (Q, R)-Policies

February 27th, 2020

Service Levels in (Q, R) Systems

p depends on multiple factors:

- Impact of stockout on future sales
- Loss of goodwill
- Corporate mindset:
  customer-focused vs. cost-focused

Service Levels in (Q, R) Systems

- When we minimize cost in the (Q, R) model, the parameter p controls how often we stock out.
- Often it is more intuitive and useful to specify a **desired minimum service level** instead of p.
- Roughly speaking, "service level" is a measure of how often we satisfy demand from in-stock inventory.

Definition of Service Level

- There are multiple ways of defining the “service level” precisely. We will use the following definition, also called the “fill rate”
- The **fill rate** is the percentage of demand that is satisfied from stock.
- Let β be the desired fill rate.

Shortage Rate

$$n(R) = \text{expected number of units short per cycle}$$
$$Q = \text{expected number of units demanded per cycle}$$

Over the long run, the fraction of demand that stocks out is

$$\text{shortage rate} = \frac{n(R)}{Q}$$

Fill Rate

The fill rate is 1 minus this amount:

$$\text{fill rate} = 1 - \text{shortage rate} = 1 - \frac{n(R)}{Q}$$

We want this to be $\geq \beta$.

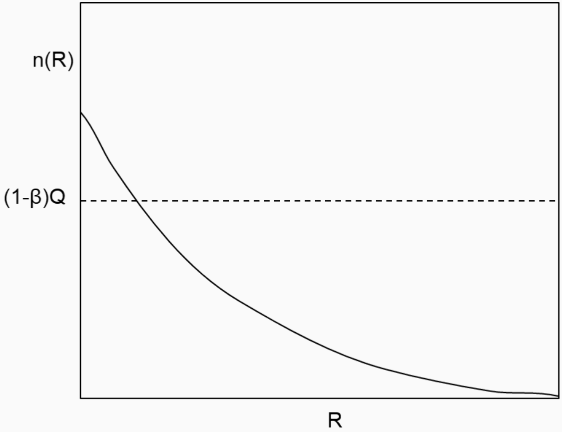
What is the fill rate in EOQ?

1. 0
2. 1
3. $\frac{1}{2}$
4. Q/λ
5. τ/T

Achieving a Fill Rate

To achieve a fill rate of β, we solve for the desired $n(R)$:

$$n(R) = (1 - \beta)Q.$$



Achieving a Fill Rate

If we have inverse of $n(R)$ we can use this inverse to calculate R, otherwise we can again use bisection search to find R such that $n(R) = (1 - \beta)Q$ (because $n(R)$ is nonincreasing in R).

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But now R does not depend on p (it now depends on β instead).

Bisection search to find R

We want to find R such that $n(R)$ is within ε of $(1-\beta)Q$.

- Set $L = 0$.
- Find an integer U large enough that $n(U) \leq (1-\beta)Q$. To do this, guess at U, check $n(R)$, and keep increasing U until $n(R) \leq (1-\beta)Q$.
- While $U - L > \varepsilon$:
    - Choose $M = (L+U)/2$
    - If $n(R) \leq (1-\beta)Q$, set $U = R$
    - If $n(R) > (1-\beta)Q$, set $L = R$

Set our final value to $R = U$

Same idea as before: iteratively finding R and Q

We will once again find an iterative procedure for finding R and Q.

We now know how to find R given a value Q (using β, not p).

How about find Q given a value of R?

Once we have R, here is how we get Q

- Previously, Q was calculated from R and the other parameters, including p:
  $Q = \sqrt{\frac{2}{h}\Big(K\lambda + p\lambda n(R)\Big)}$ (slide 54 of lecture 9)
- We now don't have p — how to get around this?

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- Previously, Q was calculated from R and the other parameters, including p:
  $Q = \sqrt{\frac{2}{h}\Big(K\lambda + p\lambda n(R)\Big)}$ (slide 54 of lecture 9)
- We now don't have p — how to get around this?
- For a given R and Q, there is an implicitly defined p. Recall:
  $F(R) = 1 - \frac{Qh}{p\lambda}$ (slide 49 of lecture 9).

Once we have R, here is how we get Q

Recall: $F(R) = 1 - \frac{Qh}{p\lambda}$ (slide 49 of lecture 9).

Solve for p:

$$p = \frac{Qh}{\lambda(1 - F(R))}$$

and use this in the equation for $Q = \sqrt{\frac{2}{h}\Big(K\lambda + p\lambda n(R)\Big)}$ (slide 54 of lecture 9):

$$Q = \sqrt{\frac{2}{h}\Big(K\lambda + \frac{Qh}{\lambda(1 - F(R))}\lambda n(R)\Big)}$$

Once we have R, here is how we get Q

$$Q = \sqrt{\frac{2}{h}\left(K\lambda + \frac{Qh}{\lambda(1-F(R))}\lambda n(R)\right)}$$

Now we have to solve for Q:

$$Q^2 = \frac{2}{h}\left(K\lambda + \frac{Qh}{\lambda(1-F(R))}\lambda n(R)\right)$$

$$Q^2 - \frac{2n(R)}{1-F(R)}Q - \frac{2K\lambda}{h} = 0$$

So

$$Q = \frac{n(R)}{1-F(R)} + \sqrt{\left(\frac{n(R)}{1-F(R)}\right)^2 + \frac{2K\lambda}{h}}.$$

Iterative Procedure for Finding Q and R for a given fill rate β

- Find $Q_0 = \text{EOQ}$
- Find R_0 from $n(R_0) = (1 - \beta)Q_0$ [using bisection search]
- Plug R_0 into the equation for Q to get Q_1.
  [use equation at bottom of previous slide. In that formula, use $n(R_0) = (1 - \beta)Q_0$.]
- Find R_1 from $n(R_1) = (1 - \beta)Q_1$ [using bisection search]
- etc.
- Stop when $R_{n+1} \approx R_n$ (stop when the R_{n+1} and R_n both give the same value when rounded to the nearest integer)

Iterative Procedure for Finding Q and R for a given fill rate β — Summary

- Lots of ugly formulas!
- Point is:
  Can start with $Q_0 = \text{EOQ}$.
  - A value for Q and β determines a value for R ($\to R_0$).
  - A value for Q and for R determines an (implicit) value for p.
  - A value for R and p determines a (new) value for Q ($\to Q_1$).
  - Rinse and repeat.

Special Case: Normally Distributed Demand

Computing Q, R policies for most demand distributions is a lot of work

You need to be able to calculate:

$F(R) = P(C(\tau) \leq R)$

and

$n(R) = E[(C(\tau) - R)^+]$

and then repeatedly use bisection search to find the value of R.

Special Case: Normally Distributed Demand

When $D_1, D_2, \dots$ are i.i.d. normally distributed with mean λ and variance σ^2:

$C(\tau)$ is distributed

1. Normal with mean $\lambda\tau$ and variance $\sigma^2\tau$
2. Normal with mean $\lambda\tau$ and variance $\sigma^2\tau^2$
3. Normal with mean $\lambda\tau^2$ and variance $\sigma^2\tau^2$
4. Poisson with mean $\lambda\tau$ and variance $\sigma^2\tau^2$
5. Poisson with mean $\lambda\tau^2$ and variance $\sigma^2\tau^2$

Special Case: Normally Distributed Demand

So we can use this for cdf of $C(\tau)$:

$$\mathbb{P}(C(\tau) \leq x) = \mathbb{P}\Big(\frac{C(\tau) - \lambda\tau}{\sigma} \leq \frac{x - \lambda\tau}{\sigma}\Big) = \Phi\Big(\frac{x - \lambda\tau}{\sigma}\Big)$$

where $\Phi(\cdot)$ is the cdf of the standard normal.

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- demand is $C(\tau)$
- inventory level is R (this was Q in Newsvendor)
- $c_u = 1$, $c_o = 0$

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Using the formula from last slide Newsvendor Lecture:

$n(R) = -\sigma\sqrt{\tau}(z\Phi(-z) - \phi(z))$

where $z = \dfrac{R - \lambda\tau}{\sigma\sqrt{\tau}}$.