Practical Tools for OR, ML and DS

Lecture 8: Inventory #2 Newsvendor

February 13th, 2020

Recall the assumptions in EOQ

- Known and constant demand rate
- Known and constant lead time
- Instantaneous receipt of material
- No quantity discounts
- No stock outs permitted
- No penalty costs (only order & holding costs)

These assumptions don’t fit in some problems

The New-York Times.

Let's consider a totally different set of assumptions

- We plan for only a single period
- Demand is random
- Deliveries are made before the demand
- Stockouts are allowed
- No holding costs
- Penalty costs are proportional to the underage and overage amounts
- No ordering costs: the cost of buying an item is accounted for by the overage penalty

Let's consider a totally different set of assumptions

- We plan for only a single period
- Demand is random
- Deliveries are made before the demand
- Stockouts are allowed
- No holding costs
- Penalty costs are proportional to the underage and overage amounts
- No ordering costs: the cost of buying an item is accounted for by the overage penalty

This set of assumptions is called the "Newsvendor" model

Newsvendor model

Notation

$$
\begin{aligned}
D &= \text{demand, a random variable} \\
F(x) &= \text{cumulative distribution function of demand} \\
c_o &= \text{penalty cost per unit of inventory remaining} \\
&\quad \text{at the end of the period, “overage” cost} \\
c_u &= \text{penalty cost per unit of unsatisfied demand,} \\
&\quad \text{“underage” cost} \\
G(Q, D) &= \text{total cost when } Q \text{ units are ordered} \\
&\quad \text{and } D \text{ is the demand}
\end{aligned}
$$

Overage

- We ordered too many!
- $\#$ of units of overage is $Q - D$ if Q is bigger than D, otherwise zero.

$$
\begin{aligned}
\# \text{ of units, overage} &= \begin{cases} Q - D & \text{for } Q \geq D \\ 0 & \text{for } Q < D \end{cases} \\
&= \max\{Q - D, 0\} \\
&= (Q - D)^+.
\end{aligned}
$$

Underage

- We ordered too few!
- $\#$ of units of overage is $D - Q$ if D is bigger than Q, otherwise zero.

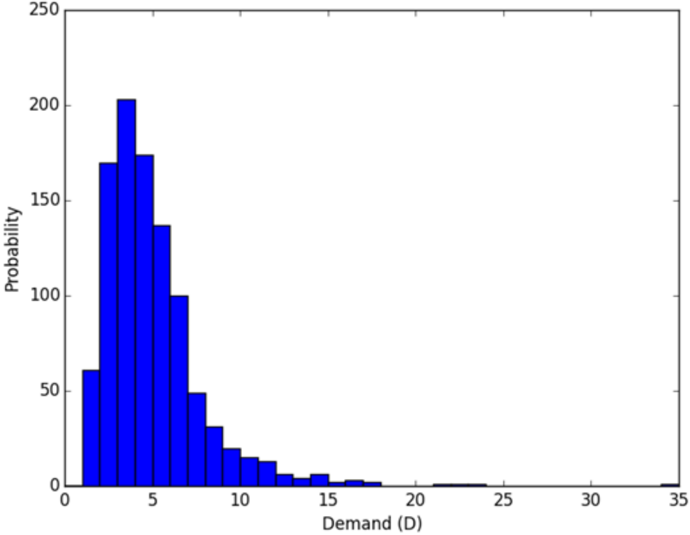
$$\begin{aligned}
\#\text{ of units, underage} &= \begin{cases} D - Q & \text{for } D \geq Q \\ 0 & \text{for } D < Q \end{cases} \\
&= \max\{D - Q, 0\} \\
&= (D - Q)^+.
\end{aligned}$$

Put this together to get the cost

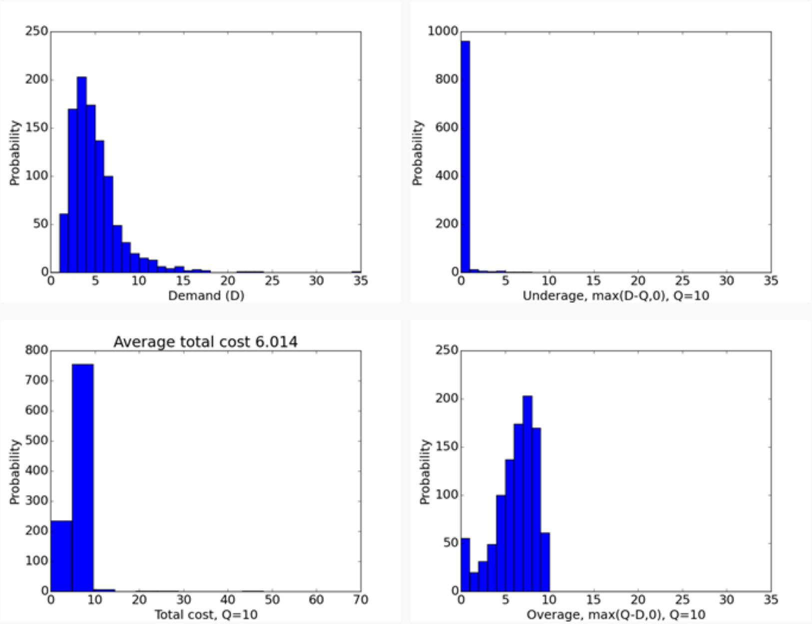
$$
\begin{aligned}
G(Q, D) &= c_o(\text{units of overage}) + c_u(\text{units of underage}) \\
&= c_o(Q - D)^+ + c_u(D - Q)^+
\end{aligned}
$$

Note: Because D is random, $G(Q, D)$ is random

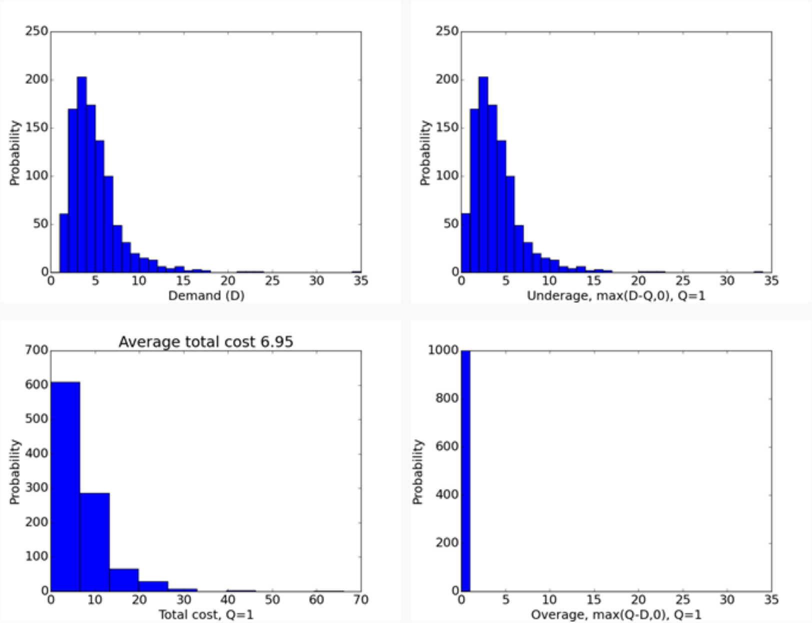
Example



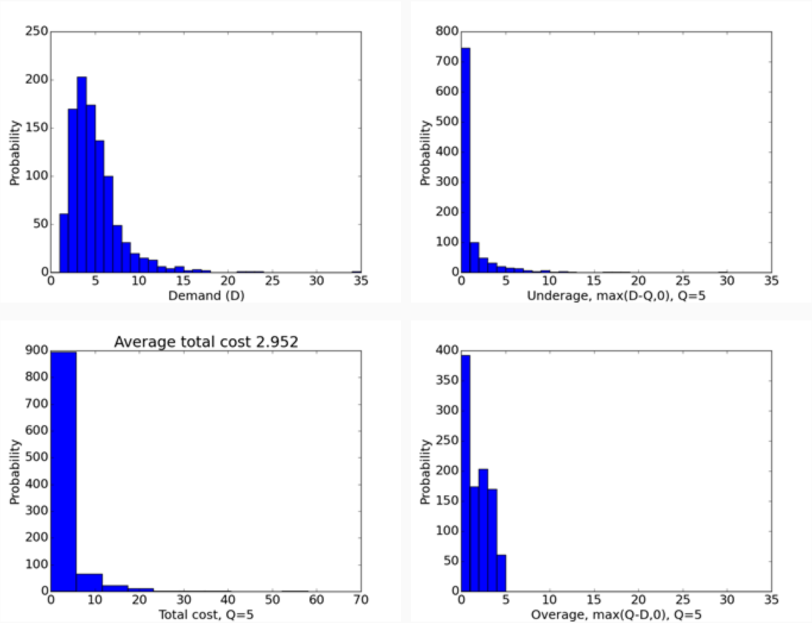
$Q = 10$ **gives an average total cost of** 6.014



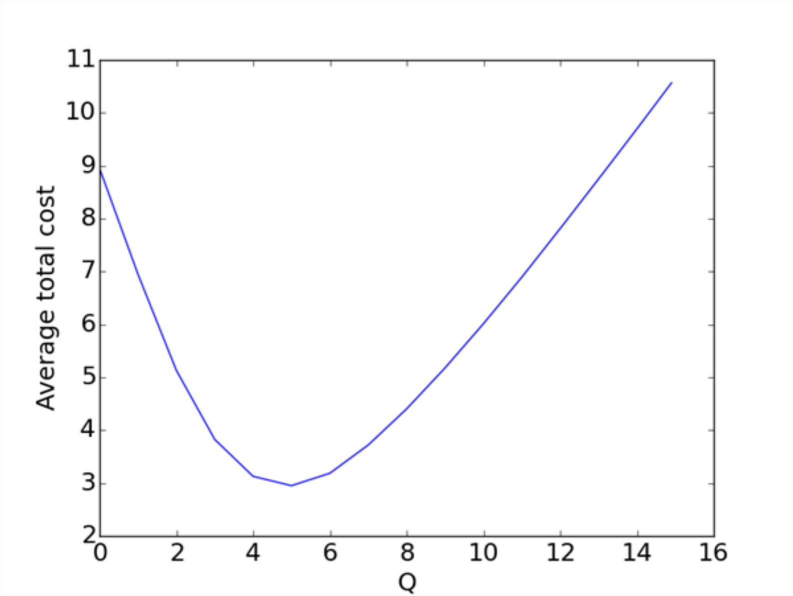
$Q = 1$ gives an average total cost of 6.95



$Q = 5$ gives an average total cost of 2.952



Average total cost vs. Q



We'll minimize the expected cost

- We don't know D when we choose Q, so we can't choose Q to minimize $G(Q, D)$.
- Instead, we'll choose Q to minimize $\mathbb{E}[G(Q, D)]$
- The expected cost is:

$$\mathbb{E}[G(Q, D)] = \mathbb{E}[c_o(Q - D)^+ + c_u(D - Q)^+]$$

- We will again find the optimum Q by setting derivative equal to zero, $\frac{d}{dQ}\mathbb{E}[G(Q, D)] = 0$, and solving for Q.

What is $\frac{d}{dQ}\mathbb{E}[G(Q,D)]$?

Let's rewrite $\mathbb{E}[G(Q,D)]$. The random variable D can be discrete or continuous.

Recall from ENGRD 2700:

- If X is a discrete random variable with probability mass function $\mathbb{P}(X = i) = p_i$ for $i = 0, 1, 2, \ldots$, then

$$\mathbb{E}g(X) = \sum_{i=0}^{\infty} g(i)p_i.$$

- If X is a continuous random variable with probability density function $f(x)$, then

$$\mathbb{E}g(X) = \int_{-\infty}^{\infty} g(x)f(x)dx.$$

What is $\frac{d}{dQ}\mathbb{E}[G(Q, D)]$?

If D is discrete then

$$\begin{aligned}\mathbb{E}[G(Q, D)] &= \mathbb{E}[c_o(Q-D)^+ + c_u(D-Q)^+] \\ &= \sum_{i=0}^{\infty}(c_o(Q-i)^+ + c_u(i-Q)^+)p_i \\ &= \sum_{i=0}^{\lfloor Q \rfloor} c_o(Q-i)p_i + \sum_{i=\lfloor Q \rfloor+1}^{\infty} c_u(i-Q)p_i.\end{aligned}$$

So, Q is not an integer, the derivative is

$$\begin{aligned}\tfrac{d}{dQ}\mathbb{E}[G(Q, D)] &= \sum_{i=0}^{\lfloor Q \rfloor} c_o p_i - \sum_{i=\lfloor Q \rfloor+1}^{\infty} c_u p_i \\ &= c_o\mathbb{P}(D < Q) - c_u\mathbb{P}(D > Q)\end{aligned}$$

What is $\frac{d}{dQ}\mathbb{E}[G(Q,D)]$?

In general,

$$
\begin{aligned}
\tfrac{d}{dQ}\mathbb{E}[G(Q,D)] &= \mathbb{E}[\tfrac{d}{dQ}G(Q,D)] \\
&= \mathbb{E}[\tfrac{d}{dQ}(c_o(Q-D)^+ + c_u(D-Q)^+)] \\
&= \mathbb{E}[\tfrac{d}{dQ}c_o(Q-D)^+ + \tfrac{d}{dQ}c_u(D-Q)^+] \\
&= \mathbb{E}[\tfrac{d}{dQ}c_o(Q-D)^+] + \mathbb{E}[\tfrac{d}{dQ}c_u(D-Q)^+],
\end{aligned}
$$

where the first equality assumes that D is discrete and Q is not an integer, or D is a continuous random variable (ensuring $G(Q,D)$ is differentiable at Q).

What is $\frac{d}{dQ}\mathbb{E}[G(Q,D)]$?

$\frac{d}{dQ}c_o(Q-D)^+ =?$

1. $-c_o$ when $Q > D$, 0 when $Q < D$, undefined when $Q = D$
2. 0 when $Q > D$, $-c_o$ when $Q < D$, undefined when $Q = D$
3. c_o when $Q > D$, 0 when $Q < D$, undefined when $Q = D$
4. 0 when $Q > D$, c_o when $Q < D$, undefined when $Q = D$
5. 0

What is $\frac{d}{dQ}\mathbb{E}[G(Q, D)]$?

$\frac{d}{dQ} c_o (Q - D)^+ =?$

1. $-c_o$ when $Q > D$, 0 when $Q < D$, undefined when $Q = D$
2. 0 when $Q > D$, $-c_o$ when $Q < D$, undefined when $Q = D$
3. **c_o when $Q > D$, 0 when $Q < D$, undefined when $Q = D$**
4. 0 when $Q > D$, c_o when $Q < D$, undefined when $Q = D$
5. 0

What is $\frac{d}{dQ}\mathbb{E}[G(Q,D)]$?

$\frac{d}{dQ} c_u(D-Q)^+ =?$

1. $-c_u$ when $Q > D$, 0 when $Q < D$, undefined when $Q = D$
2. 0 when $Q > D$, $-c_u$ when $Q < D$, undefined when $Q = D$
3. c_u when $Q > D$, 0 when $Q < D$, undefined when $Q = D$
4. 0 when $Q > D$, c_u when $Q < D$, undefined when $Q = D$
5. 0

What is $\frac{d}{dQ}\mathbb{E}[G(Q, D)]$?

$\frac{d}{dQ} c_u (D - Q)^+ =?$

1. $-c_u$ when $Q > D$, 0 when $Q < D$, undefined when $Q = D$
2. 0 **when** $Q > D$**,** $-c_u$ **when** $Q < D$**, undefined when** $Q = D$
3. c_u when $Q > D$, 0 when $Q < D$, undefined when $Q = D$
4. 0 when $Q > D$, c_u when $Q < D$, undefined when $Q = D$
5. 0

What is $\frac{d}{dQ}\mathbb{E}[G(Q,D)]$?

$$
\begin{aligned}
\frac{d}{dQ}\mathbb{E}[G(Q,D)] &= \mathbb{E}[\tfrac{d}{dQ}c_o(Q-D)^+] + \mathbb{E}[\tfrac{d}{dQ}c_u(D-Q)^+] \\
&= \mathbb{E}[c_o\mathbb{1}_{\{Q>D\}}] + \mathbb{E}[-c_u\mathbb{1}_{\{Q<D\}}] \\
&= c_o\mathbb{E}[\mathbb{1}_{\{Q>D\}}] - c_u\mathbb{E}[\mathbb{1}_{\{Q<D\}}]
\end{aligned}
$$

Note: $\mathbb{1}_{\{Q>D\}}$ means "1 when $Q > D$, and 0 otherwise".

What is $\mathbb{E}[\mathbb{1}_{\{Q>D\}}]$?

1. $\mathbb{P}(Q < D)$
2. $\mathbb{P}(Q > D)$
3. $\mathbb{E}[Q]$
4. $\mathbb{E}[Q - D]$
5. None of the above

Note: $\mathbb{1}_{\{Q>D\}}$ means "1 when $Q > D$, and 0 otherwise".

What is $\frac{d}{dQ}\mathbb{E}[G(Q,D)]$?

In general,

$$\frac{d}{dQ}\mathbb{E}[G(Q,D)] = c_o\mathbb{E}[\mathbb{1}_{\{Q>D\}}] - c_u\mathbb{E}[\mathbb{1}_{\{Q<D\}}]$$

What is $\frac{d}{dQ}\mathbb{E}[G(Q,D)]$?

In general,

$$\begin{aligned}\frac{d}{dQ}\mathbb{E}[G(Q,D)] &= c_o\mathbb{E}[\mathbb{1}_{\{Q>D\}}] - c_u\mathbb{E}[\mathbb{1}_{\{Q<D\}}] \\ &= c_o\mathbb{P}(Q>D) - c_u\mathbb{P}(Q<D)\end{aligned}$$

What is $\frac{d}{dQ}\mathbb{E}[G(Q,D)]$?

In general,

$$\begin{aligned}
\frac{d}{dQ}\mathbb{E}[G(Q,D)] &= c_o\mathbb{E}[\mathbb{1}_{\{Q>D\}}] - c_u\mathbb{E}[\mathbb{1}_{\{Q<D\}}] \\
&= c_o\mathbb{P}(Q>D) - c_u\mathbb{P}(Q<D) \\
&= c_o\mathbb{P}(D<Q) - c_u(1-\mathbb{P}(D\le Q)) \\
&= c_o\mathbb{P}(D\le Q) - c_u(1-\mathbb{P}(D\le Q)) \\
&= -c_u + (c_u+c_o)\mathbb{P}(D\le Q)
\end{aligned}$$

where the penultimate inequality assumes D is a continuous random variable or D is a discrete random variable taking integer values and Q is not an integer.

Setting $\frac{d}{dQ}\mathbb{E}[G(Q,D)] = 0$

$$\frac{d}{dQ}\mathbb{E}[G(Q,D)] = -c_u + (c_u + c_o)\mathbb{P}(D \leq Q) = 0$$

means we want Q^* so that

$$\mathbb{P}(D \leq Q^*) = \frac{c_u}{c_u + c_o}.$$

Setting $\frac{d}{dQ}\mathbb{E}[G(Q,D)] = 0$

$$\frac{d}{dQ}\mathbb{E}[G(Q,D)] = -c_u + (c_u + c_o)\mathbb{P}(D \le Q) = 0$$

means we want Q^* so that

$$\mathbb{P}(D \le Q^*) = \frac{c_u}{c_u + c_o}.$$

Does this Q^* give us *minimum* cost?

Setting $\frac{d}{dQ}\mathbb{E}[G(Q,D)] = 0$

For $Q < Q^*$:

$$\begin{aligned}\frac{d}{dQ}\mathbb{E}[G(Q,D)] &= -c_u + (c_u + c_o)\mathbb{P}(D \leq Q) \\ &< -c_u + (c_u + c_o)\mathbb{P}(D \leq Q^*) = 0\end{aligned}$$

So costs are decreasing in Q until Q^*!

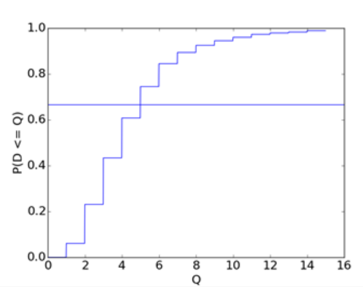
Setting $\frac{d}{dQ}\mathbb{E}[G(Q,D)] = 0$

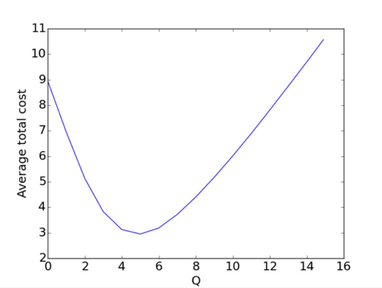
For $Q > Q^*$:

$$\begin{aligned}\frac{d}{dQ}\mathbb{E}[G(Q,D)] &= -c_u + (c_u + c_o)\mathbb{P}(D \leq Q) \\ &> -c_u + (c_u + c_o)\mathbb{P}(D \leq Q^*) = 0\end{aligned}$$

So costs are increasing in Q after Q^*!

Pictorially





How to find Q^*?

$$\mathbb{P}(D \leq Q^*) = \frac{c_u}{c_u + c_o}.$$

In words: We want Q^* so that the probability that the demand is Q^* or less is $\frac{c_u}{c_u + c_o}$.

If $F(\cdot)$ is cdf of D, then in terms of cdf:

$$F(Q^*) = \frac{c_u}{c_u + c_o}.$$

How to find Q^*?

If $F(\cdot)$ has an *inverse* $F^{-1}(\cdot)$ then

$$Q^* = F^{-1}(\frac{c_u}{c_u + c_o}).$$

How to find Q^*?

If $F(\cdot)$ has an *inverse* $F^{-1}(\cdot)$ then

$$Q^* = F^{-1}(\frac{c_u}{c_u + c_o}).$$

Continuous random variables have cdfs that are invertible!

$F^{-1}(\cdot)$ and quantiles

- $F^{-1}(q)$ is called the "q-quantile" (of the random variable that has cdf $F(\cdot)$)
- Excel, Python, and R implement the inverse cumulative distribution function for many common distributions
- Here are some useful functions in Excel:
    - `NORMAL.INV(probability, m, v)`
    - `LOGNORM.INV(probability, m, s)`
    - `GAMMA.INV(probability, alpha, beta)`

How to find Q^* without an inverse?

$$\mathbb{P}(D \leq Q^*) = \frac{c_u}{c_u + c_o}.$$

You can use bisection search to find Q^* (upto some ε):

- Set $L = 0$ and find an integer U large enough that

$$\mathbb{P}(D \leq U) \leq \frac{c_u}{c_u + c_o}.$$

How to find Q^* without an inverse?

$$\mathbb{P}(D \leq Q^*) = \frac{c_u}{c_u + c_o}.$$

You can use bisection search to find Q^* (upto some ε):

- Set $L = 0$ and find an integer U large enough that

$$\mathbb{P}(D \leq U) \leq \frac{c_u}{c_u + c_o}.$$

  To do this, guess $U = 1$, check $\mathbb{P}(D \leq U)$, and keep doubling U until $\mathbb{P}(D \leq U) \geq \frac{c_u}{c_u + c_o}$.

How to find Q^* without an inverse?

$$\mathbb{P}(D \leq Q^*) = \frac{c_u}{c_u + c_o}.$$

You can use bisection search to find Q^* (upto some ε):

- Set $L = 0$ and find an integer U large enough that

$$\mathbb{P}(D \leq U) \leq \frac{c_u}{c_u + c_o}.$$

- While $U - L > \varepsilon$:
  - Choose $M = \frac{L + U}{2}$.
  - If $\mathbb{P}(D \leq M) \geq \frac{c_u}{c_u + c_o}$, set $U = M$
  - If $\mathbb{P}(D \leq M) < \frac{c_u}{c_u + c_o}$, set $L = M$.

Example

- Lowe's sells holiday lights for the winter holiday season. During the holiday season, the lights sell for \$2.00 each.
- Since the product is seasonal, the store decides to sell all unsold lights during the January clearance for \$0.50 each.
- Each string of lights costs the store \$1.
- Past demand has followed a log-normal(7,3) distribution, which means that the natural log of demand is normal with mean 7 and standard deviation 3.
- Find the optimal order quantity for the season.

Example

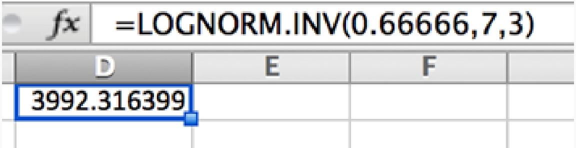
We see that if Lowe's orders too many, the cost is \$0.50 each. If they order too few, each lost sale represents \$1 of unrealized profit. Thus, $c_o = \$0.50$, and $c_u = \$1$. Hence we want Q so that:

$$F(Q) = \frac{c_u}{c_u + c_o} = \frac{1}{1 + .50} = 0.6667$$

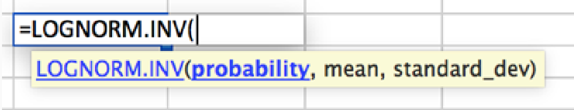
where F is the cdf of the log-normal(7,3) distribution.

Example

- Excel's LOGNORM.INV function tells us:



- The syntax of LOGNORM.INV is:



- So, we should stock 3992 holiday lights

Newsvendor with the Normal Distribution

If you have a computer:

- Suppose demand is $D \sim$Normal(μ, σ^2)
- The optimal order quantity Q^* is $F^{-1}(c_u/(c_u + c_o))$
- In Excel, we can calculate this via `NORM.INV`

Newsvendor with the Normal Distribution

If you don't have a computer:

We'll now show you how to compute Q^* and the expected cost $\mathbb{E}[G(Q, D)]$ when $D \sim \text{Normal}(\mu, \sigma^2)$ without a computer using pencil/paper and normal distribution tables.

Normal Distribution Tables

Normal Distribution Tables

File Edit View Insert Format Data Tools Add-ons Help Last edit was on January 25

Table values are F(z)=P(Z<=z), where z is the value at left plus the value at top, e.g. P(Z<=3.99) = .99997

Z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147

Computing Q^*

- $Q^* = F^{-1}(c_u/(c_u + c_o))$

$$\begin{aligned} c_u/(c_u + c_o) &= \mathbb{P}(D \leq Q^*) \\ &= \mathbb{P}(\mu + \sigma Z \leq Q^*) \text{ where } Z \sim N(0,1) \\ &= \mathbb{P}(Z \leq (Q^* - \mu)/\sigma) \end{aligned}$$

- Step 1: Find z^* such that $c_u/(c_u + c_o) = \mathbb{P}(Z \leq z^*)$ from the normal distribution tables
  Step 2: $Q^* = \mu + \sigma z^*$

Computing the expected cost if D is normal

$$
\begin{aligned}
\mathbb{E}[G(Q,D)] &= \mathbb{E}[c_o(Q-D)^+ + c_u(D-Q)^+] \\
&= c_o \int_{-\infty}^{Q} (Q-x)f(x)dx + c_u \int_{Q}^{\infty} (x-Q)f(x)dx.
\end{aligned}
$$

Let's try and calculate $\int_{-\infty}^{Q} (Q-x)f(x)dx$.

Computing the expected cost if D is normal

Transform to standard normal: $y = (x - \mu)/\sigma$

$$\int_{-\infty}^{Q} (Q - x) f(x) dx = \int_{-\infty}^{(Q-\mu)/\sigma} (Q - (\sigma y + \mu)) \varphi(y) dy$$

$$= \underbrace{\int_{-\infty}^{(Q-\mu)/\sigma} (Q - \mu) \varphi(y) dy}_{(Q-\mu)\mathbb{P}(D \leq Q)} - \sigma \int_{-\infty}^{(Q-\mu)/\sigma} y \varphi(y) dy.$$

Computing the expected cost if D is normal

- What about $\int_{-\infty}^{t} x\varphi(x)dx$?
- $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$
- Here is a weird little fact:
  $\frac{d}{dx}\varphi(x) = -x\frac{1}{\sqrt{2\pi}} e^{-x^2/2} = -x\varphi(x)!$

$$\text{So: } \int_{-\infty}^{t} x\varphi(x)dx = -\int_{-\infty}^{t} (\tfrac{d}{dx}\varphi(x))dx$$
$$= -(\varphi(t) - \varphi(-\infty)) = -\varphi(t)!$$

Computing the expected cost if D is normal

$$\int_{-\infty}^{Q}(Q-x)f(x)dx = \int_{-\infty}^{(Q-\mu)/\sigma}(Q-(\sigma x+\mu))\varphi(x)dx$$

$$= \underbrace{\int_{-\infty}^{(Q-\mu)/\sigma}(Q-\mu)\varphi(x)dx}_{(Q-\mu)\mathbb{P}(D\leq Q)} - \sigma \underbrace{\int_{-\infty}^{(Q-\mu)/\sigma} x\varphi(x)dx}_{-\varphi((Q-\mu)/\sigma)}.$$

Computing the expected cost if D is normal

$$
\begin{aligned}
\int_{-\infty}^{Q} (Q-x)f(x)dx &= (Q-\mu)\mathbb{P}(D \le Q) + \sigma\varphi((Q-\mu)/\sigma) \\
&= \sigma(z\mathbb{P}(D \le Q) + \varphi(z)) \\
&= \sigma(z\Phi(z) + \varphi(z))
\end{aligned}
$$

where $z = (Q-\mu)/\sigma$.

Computing the expected cost if D is normal

$$\begin{aligned}\mathbb{E}[G(Q,D)] &= c_o \int_{-\infty}^{Q} (Q-x) f(x) dx + c_u \int_{Q}^{\infty} (x-Q) f(x) dx \\ &= c_o \sigma (z\Phi(z) + \varphi(z)) + c_u \int_{Q}^{\infty} (x-Q) f(x) dx.\end{aligned}$$

What about $\int_{Q}^{\infty} (x-Q) f(x) dx$?

Computing the expected cost if D is normal

Let $\tilde{D} = -D$, a normally distributed random variable with mean $\tilde{\mu} = -\mu$ and variance σ^2.

Now

$$\int_Q^\infty (x - Q) f(x) dx = \int_{-\infty}^{-Q} (-x - Q) \tilde{f}(x) dx$$
$$= \int_{-\infty}^{\tilde{Q}} (\tilde{Q} - x) \tilde{f}(x).$$

Can use previous result:

$$\int_{-\infty}^{\tilde{Q}} (\tilde{Q} - x) \tilde{f}(x) dx = \sigma(\tilde{z}\Phi(\tilde{z}) + \varphi(\tilde{z})).$$

Computing the expected cost if D is normal

$$
\begin{aligned}
\mathbb{E}[G(Q,D)] &= c_o \int_{-\infty}^{Q} (Q-x) f(x) dx + c_u \int_{Q}^{\infty} (x-Q) f(x) dx \\
&= c_o \sigma (z\Phi(z) + \varphi(z)) + c_u \sigma (\tilde{z}\Phi(\tilde{z}) + \varphi(\tilde{z})).
\end{aligned}
$$

where $z = (Q-\mu)/\sigma$ and $\tilde{z} = (\tilde{Q} - \tilde{\mu})/\sigma$.

Computing the expected cost if D is normal

$$\mathbb{E}[G(Q,D)] = c_o \int_{-\infty}^{Q} (Q-x)f(x)dx + c_u \int_{Q}^{\infty} (x-Q)f(x)dx$$
$$= c_o\sigma(z\Phi(z)+\varphi(z)) + c_u\sigma(\tilde{z}\Phi(\tilde{z})+\varphi(\tilde{z})).$$

where $z = (Q-\mu)/\sigma$ and $\tilde{z} = (\tilde{Q}-\tilde{\mu})/\sigma$.

Note: $\tilde{z} = (\tilde{Q}-\tilde{\mu})/\sigma = (-Q-(-\mu))/\sigma = (\mu - Q)/\sigma = -z$.

Computing the expected cost if D is normal

$$\begin{aligned}\mathbb{E}[G(Q,D)] &= c_o\sigma(z\Phi(z)+\varphi(z)) + c_u\sigma(\tilde{z}\Phi(\tilde{z})+\varphi(\tilde{z}))\\ &= c_o\sigma(z\Phi(z)+\varphi(z)) + c_u\sigma(-z\Phi(-z)+\varphi(-z)).\end{aligned}$$

where $z = (Q-\mu)/\sigma$.

Computing the expected cost if D is normal

$$\begin{aligned}\mathbb{E}[G(Q,D)] &= c_o\sigma(z\Phi(z)+\varphi(z)) + c_u\sigma(\tilde{z}\Phi(\tilde{z})+\varphi(\tilde{z}))\\ &= c_o\sigma(z\Phi(z)+\varphi(z)) + c_u\sigma(-z\Phi(-z)+\varphi(-z)).\end{aligned}$$

where $z = (Q-\mu)/\sigma$.

Finally note that $\varphi(\tilde{z}) = \varphi(-z) = \varphi(z)$ because φ is symmetric about 0.

Computing the expected cost if D is normal

$$
\begin{aligned}
\mathbb{E}[G(Q, D)] &= c_o\sigma(z\Phi(z) + \varphi(z)) + c_u\sigma(-z\Phi(-z) + \varphi(-z)) \\
&= c_o\sigma(z\Phi(z) + \varphi(z)) - c_u\sigma(z\Phi(-z) - \varphi(z))
\end{aligned}
$$

where $z = (Q - \mu)/\sigma$.