Practical Tools for OR, ML and DS

Lecture 8: Inventory #1 Introduction and EOQ

February 11th, 2020; updated slides

How should you buy plastic resin for your plastics factory?

80,000 pounds at
$0.68 per pound

1,000 pounds at
$0.88 per pound

How should you buy garments to stock in your store?

Why hold inventory?

- Uncertainties in external demand
- Production smoothing
- Speculation
- Inventory in transit, or in shipping/receiving
- Volume discounts from suppliers

Goal: keeping costs low

We usually consider 3 kinds of costs when managing inventory:

- Holding costs
- Order costs
- Penalty costs

Holding Costs

- All costs incurred to keep items in inventory
- Warehouse rent, taxes, insurance, climate control, lost interest on tied-up capital, etc.
- Usually written using the notation "h" which is the holding cost to store one item in inventory over one time unit
- Units are often \$/units inventory-unit time

Order Costs

Usually has a fixed component and a variable component:

$$C(x) = K + cx$$

- K = fixed costs per order
- c = variable order cost
- x = number of units in order

Fixed costs can include bookkeeping, order generation, receiving & handling

Penalty Cost

Cost of not having sufficient stock on hand

- Backorder costs
  Bookkeeping, discount price, extra shipping costs, loss of customer goodwill
- Lost sales costs
  Lost profit, as potential customer went elsewhere

$p =$ cost per unit backordered

We won't consider the length of the backorder

Three theoretical approaches to managing inventory in this course

- Economic Order Quantity (EOQ)
- Newsvendor
- Continuous Review

Each makes different assumptions, and is a good choice when those assumptions are approximately met

Three theoretical approaches to managing inventory in this course

- Economic Order Quantity (EOQ)
- Newsvendor
- Continuous Review

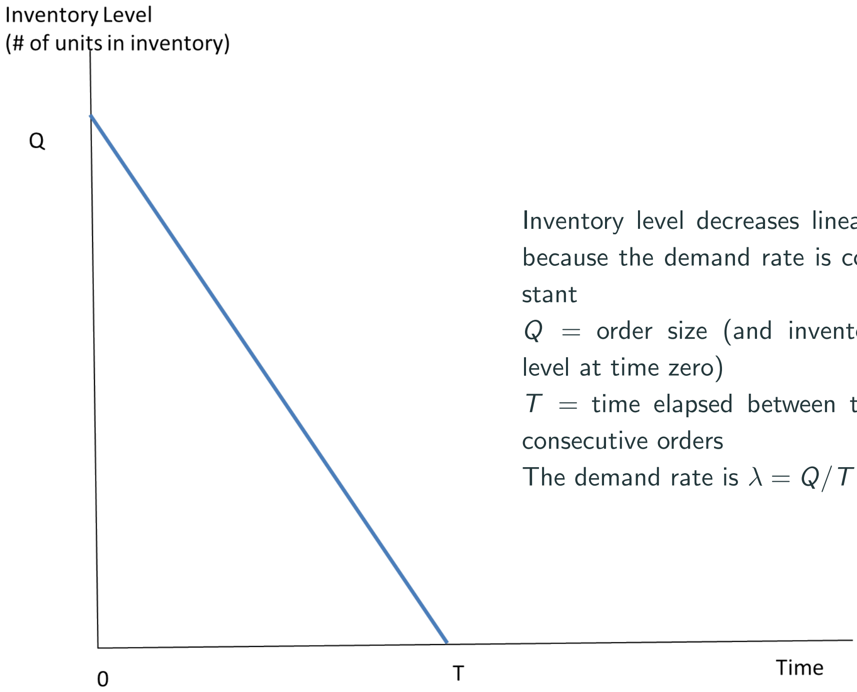
Each makes different assumptions, and is a good choice when those assumptions are approximately met

We'll also talk about using data + simulation to apply these approaches

Economic Order Quantity (EOQ)

Assumptions

- Known and constant **demand rate**
- Known and constant **lead time**
- Instantaneous receipt of material
- No quantity discounts
- No stock outs permitted
- No penalty costs (only order & holding costs)



Inventory level decreases linearly because the demand rate is constant

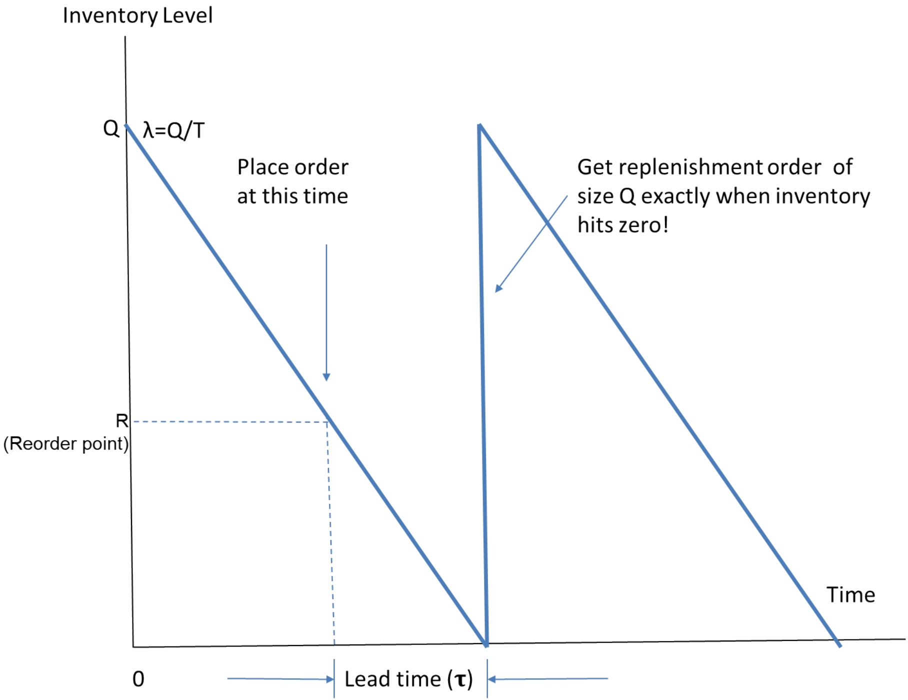
Q = order size (and inventory level at time zero)

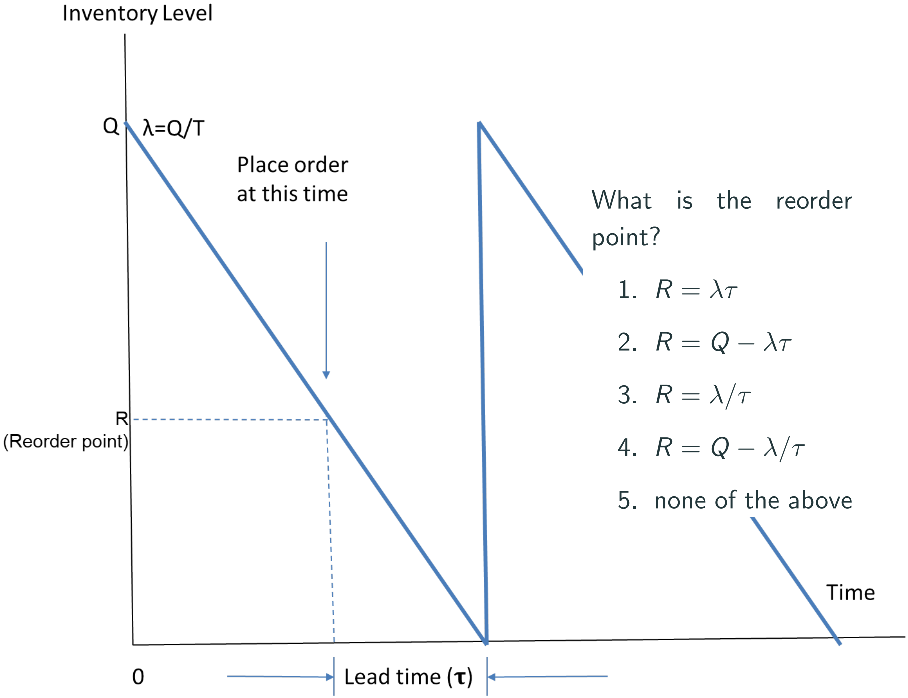
T = time elapsed between two consecutive orders

The demand rate is $\lambda = Q/T$

When should we order more?

- We need to plan ahead. There is a lead time between when we place the order and when we receive it
- Re-order point: when the inventory hits that level, we should place an order (we call this a "replenishment order")
- A good time to place the order is when we are one "lead time" away from zero inventory

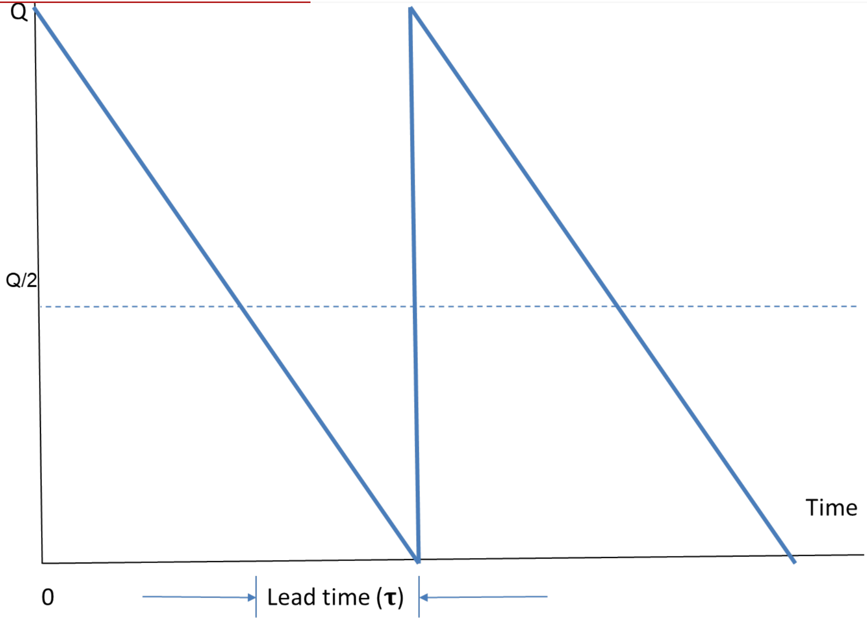

Inventory Level
Q
λ=Q/T
Place order
at this time
Get replenishment order of
size Q exactly when inventory
hits zero!
R
(Reorder point)
Time
0
Lead time (τ)



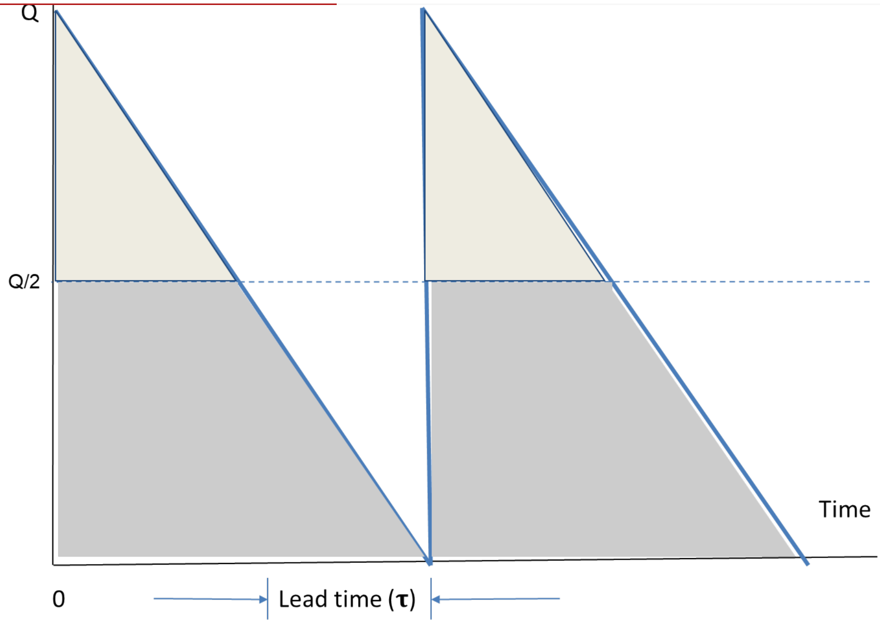
What is the reorder point?

1. $R = \lambda\tau$
2. $R = Q - \lambda\tau$
3. $R = \lambda/\tau$
4. $R = Q - \lambda/\tau$
5. none of the above

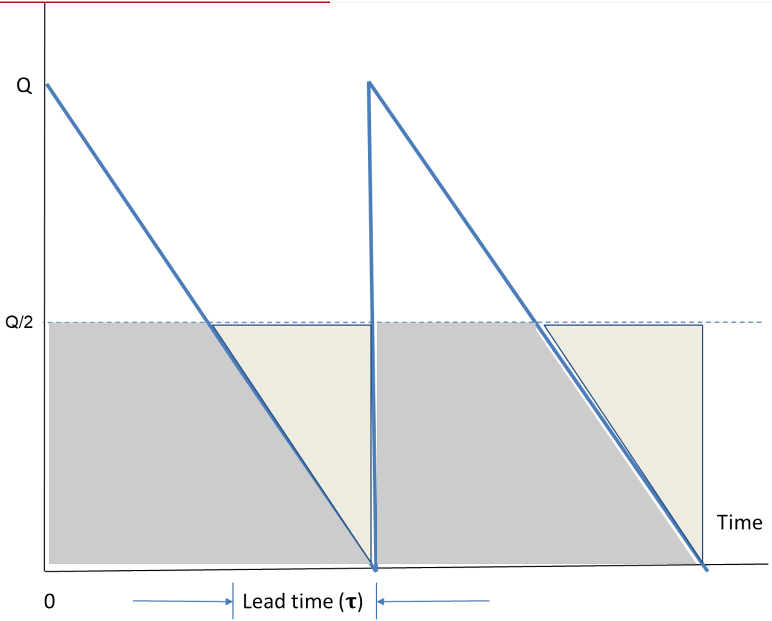
The average inventory level is $Q/2$



The average inventory level is $Q/2$



The average inventory level is $Q/2$



Knowing the average inventory level tells us the average holding cost

- Average inventory level per unit time $= Q/2$
- Therefore average holding cost per unit time

Knowing the average inventory level tells us the average holding cost

- Average inventory level per unit time $= Q/2$
- Therefore average holding cost per unit time $= hQ/2$

We can also calculate after how many units of time we need to order

Order size is Q

Demand rate is λ items per unit time

Therefore we need to order every $T =$

We can also calculate after how many units of time we need to order

Order size is Q

Demand rate is λ items per unit time

Therefore we need to order every $T = Q/\lambda$ time units.

Cost per unit time

Holding cost: h dollars per unit of inventory per unit time

Average inventory level per unit time $= Q/2$

Average holding cost per unit time:

Cost per unit time

Holding cost: h dollars per unit of inventory per unit time

Average inventory level per unit time $= Q/2$

Average holding cost per unit time: $hQ/2$

Cost per unit time

Holding cost: h dollars per unit of inventory per unit time

Average inventory level per unit time $= Q/2$

Average holding cost per unit time: $hQ/2$

Ordering cost: fixed cost K, variable cost c (per item)

Ordering cost per unit time:

Cost per unit time

Holding cost: h dollars per unit of inventory per unit time

Average inventory level per unit time $= Q/2$

Average holding cost per unit time: $hQ/2$

Ordering cost: fixed cost K, variable cost c (per item)

Ordering cost per unit time: $(K + cQ)/T$

Cost per unit time

Holding cost: h dollars per unit of inventory per unit time

Average inventory level per unit time $= Q/2$

Average holding cost per unit time: $hQ/2$

Ordering cost: fixed cost K, variable cost c (per item)

Ordering cost per unit time: $(K + cQ)/T$
$= (K + cQ)/(Q/\lambda) = \lambda K/Q + c\lambda$

Cost per unit time

Holding cost: h dollars per unit of inventory per unit time

Average inventory level per unit time $= Q/2$

Average holding cost per unit time: $hQ/2$

Ordering cost: fixed cost K, variable cost c (per item)

Ordering cost per unit time: $(K + cQ)/T$
$= (K + cQ)/(Q/\lambda) = \lambda K/Q + c\lambda$

So (average) total cost per unit time:
$G = hQ/2 + \lambda K/Q + c\lambda.$

Cost per unit time

Holding cost: h dollars per unit of inventory per unit time

Average inventory level per unit time $= Q/2$

Average holding cost per unit time: $hQ/2$

Ordering cost: fixed cost K, variable cost c (per item)

Ordering cost per unit time: $(K + cQ)/T$
$= (K + cQ)/(Q/\lambda) = \lambda K/Q + c\lambda$

So (average) total cost per unit time:
$G = hQ/2 + \lambda K/Q + c\lambda$. This is what we want to minimize

Recall from Calculus

$$G(Q) = hQ/2 + \lambda K/Q + c\lambda.$$

We have a continuous function in one variable (h, λ, K and c are parameters that are given to us).

To find optimum:

Recall from Calculus

$$G(Q) = hQ/2 + \lambda K/Q + c\lambda.$$

We have a continuous function in one variable (h, λ, K and c are parameters that are given to us).

To find optimum:

Find critical points. Check critical points (and boundary) to see which is minimizer.

Recall from Calculus

$$G(Q) = hQ/2 + \lambda K/Q + c\lambda.$$

We have a continuous function in one variable (h, λ, K and c are parameters that are given to us).

To find optimum:

Find critical points. Check critical points (and boundary) to see which is minimizer.

Critical points are where derivative is zero.

Economic Order Quantity Q^*

$$G(Q) = hQ/2 + \lambda K/Q + c\lambda.$$

Critical points: point where Q is so that $\frac{d}{dQ}G(Q) = 0$.

Economic Order Quantity Q^*

$$G(Q) = hQ/2 + \lambda K/Q + c\lambda.$$

Critical points: point where Q is so that $\frac{d}{dQ}G(Q) = 0$.

What is $\frac{d}{dQ}G(Q)$?

Economic Order Quantity Q^*

$$\frac{d}{dQ}G(Q) = h/2 - \lambda K/Q^2$$

So critical point (solve for Q):

$$h/2 - \lambda K/Q^2 = 0$$

$$Q^2 = 2\lambda K/h$$

$$Q^* = \sqrt{2\lambda K/h}$$

(because we can only order nonnegative amounts.)

Economic Order Quantity Q^*

$$G(Q) = hQ/2 + \lambda K/Q + c\lambda.$$

$$Q^* = \sqrt{2\lambda K/h}$$

Is this really minimizer?

Economic Order Quantity Q^*

$$G(Q) = hQ/2 + \lambda K/Q + c\lambda.$$

$$Q^* = \sqrt{2\lambda K/h}$$

Is this really minimizer?

Yes! $\lim_{Q \to 0} G(Q) = \infty$, and $\lim_{Q \to \infty} G(Q) = \infty$. (Could also check 2nd derivative.)

There's a tradeoff between order frequency & inventory level

- If we order large amounts infrequently, we'll have a high average inventory level.
- Our order costs per year will be low, but our holding costs will be high
- Once we turn cash into inventory, we'll have to sell it to turn it into cash again

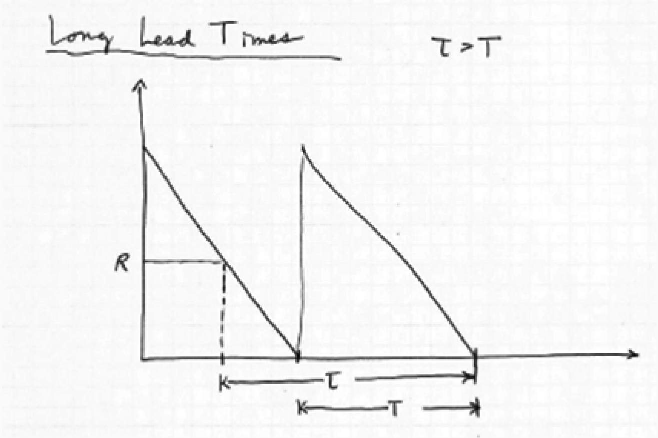
Large inventories are risky

Risks:

- Product is a fad, and stops being popular
- Product becomes obsolete, and stops selling
- A recession hits, and consumers stop spending
- The product doesn't sell as well as predicted

Long lead times

Be careful of long lead times, then you might have to place a replenishment order well ahead, as shown:



Cost is not too sensitive to Q

- The curve for G is shallow, so even if your lot size or order quantity is off from the optimal, you won't do too badly
- There are many reasons you might want to place a non-optimal order (maybe the product comes in boxes of 24 units...)

Cost is not too sensitive to Q

Let's see if we can make this precise, by finding a nice upper bound for $G(Q)/G(Q^*)$.

Reinterpreting Q^*

$$G(Q) = hQ/2 + \lambda K/Q + c\lambda.$$

Q^* is so that

$$h/2 - \lambda K/Q^2 = 0.$$

Reinterpreting Q^*

$$G(Q) = hQ/2 + \lambda K/Q + c\lambda.$$

Q^* is so that

$$h/2 - \lambda K/Q^2 = 0.$$

Let's reinterpret that. Multiply Q:

$$hQ/2 - \lambda K/Q = 0.$$

What does this say?

Reinterpreting Q^*

Q^* is so that

$$hQ/2 = \lambda K/Q.$$

Q^* is so that the holding cost per unit time and the fixed ordering costs are equal!

Total Cost of Non-Optimal Q

$$G(Q) = hQ/2 + \lambda K/Q + c\lambda.$$

$$G(Q^*) = hQ^*/2 + \lambda K/Q^* + c\lambda = hQ^* + c\lambda = 2\lambda K/Q^* + c\lambda$$

When $c = 0$, the ratio $G(Q)/G^*(Q)$:

$$\begin{aligned}
\frac{G(Q)}{G(Q^*)} &= \frac{hQ/2 + \lambda K/Q}{G(Q^*)} \\
&= \frac{hQ/2}{G(Q^*)} + \frac{\lambda K/Q}{G(Q^*)} \\
&= \frac{hQ/2}{hQ^*} + \frac{\lambda K/Q}{2\lambda K/Q^*} \\
&\le \tfrac{1}{2}\frac{Q}{Q^*} + \tfrac{1}{2}\frac{Q^*}{Q}
\end{aligned}$$

Total Cost of Non-Optimal Q

Now note that $(a+t)/(b+t)$ is nonincreasing in t if $a \geq b$ (you can check this by taking a derivative; intuitively it makes sense because $(a+t)/(b+t) \to 1$ as $t \to \infty$ and $a/b \geq 1$).

We therefore have

$$\frac{G(Q)}{G(Q*)} \leq \frac{\tilde{G}(Q)}{\tilde{G}(Q^*)} = \tfrac{1}{2}\frac{Q}{Q^*} + \tfrac{1}{2}\frac{Q^*}{Q}.$$

Ordering intervals

Also $T^* = Q^*/\lambda$ and $T = Q/\lambda$.

Therefore

$$\frac{T}{T^*} = \frac{Q}{Q^*}.$$

Ordering intervals

Also $T^* = Q^*/\lambda$ and $T = Q/\lambda$.

Therefore

$$\frac{T}{T^*} = \frac{Q}{Q^*}.$$

So

$$\begin{aligned}\frac{G(Q)}{G(Q^*)} &\leq \tfrac{1}{2}\frac{Q}{Q^*} + \tfrac{1}{2}\frac{Q^*}{Q} \\ &= \tfrac{1}{2}\frac{T}{T^*} + \tfrac{1}{2}\frac{T^*}{T}.\end{aligned}$$

Power of Two Ordering

If we have to place orders every week, or every other week, or every fourth week, then our choices are

- Weekly $= 2^0$
- Bi-weekly $= 2^1$
- Every four weeks $= 2^2$.

Power of Two Ordering

The worst isn't that bad

Suppose we evaluate the cost for $T = 2^m$ for all m, and select the best one.

How bad can our total cost per time unit be, compare to the cost of Q^*?

The worst isn't that bad

Suppose we evaluate the cost for $T = 2^m$ for all m, and select the best one.

How bad can our total cost per time unit be, compare to the cost of Q^*?

Can we give a simple (constant, independent of T^*) upper bound on

$$\frac{G(Q)}{G(Q^*)} \leq \min_i \left\{ \tfrac{1}{2} \frac{2^i}{T^*} + \tfrac{1}{2} \frac{T^*}{2^i} \right\}$$

for any T^*?

The worst isn't that bad

The worst isn't that bad

Let t be so that $2^t \leq T^* \leq 2^{t+1}$.

Then

$$\frac{G(Q)}{G(Q^*)} \leq \min\{\tfrac{1}{2}\frac{2^t}{T^*} + \tfrac{1}{2}\frac{T^*}{2^t}, \tfrac{1}{2}\frac{2^{t+1}}{T^*} + \tfrac{1}{2}\frac{T^*}{2^{t+1}}\}.$$

Now want to find worst possible T^* to find upper bound.

The worst isn't that bad

$$\frac{d}{dT^*}\left[\frac{2^t}{T^*} + \frac{T^*}{2^t}\right] = -\frac{2^t}{(T^*)^2} + \frac{1}{2^t} > 0 \text{ for } T^* > 2^t.$$

$$\frac{d}{dT^*}\left[\frac{2^{t+1}}{T^*} + \frac{T^*}{2^{t+1}}\right] = -\frac{2^{t+1}}{(T^*)^2} + \frac{1}{2^{t+1}} < 0 \text{ for } T^* < 2^{t+1}.$$

So worst possible T^* is where these are equal:

$$-\frac{2^t}{(T^*)^2} + \frac{1}{2^t} = -\frac{2^{t+1}}{(T^*)^2} + \frac{1}{2^{t+1}}.$$

The worst isn't that bad

$$\frac{d}{dT^*}\left[\frac{2^t}{T^*}+\frac{T^*}{2^t}\right]=-\frac{2^t}{(T^*)^2}+\frac{1}{2^t}>0 \text{ for } T^*>2^t.$$

$$\frac{d}{dT^*}\left[\frac{2^{t+1}}{T^*}+\frac{T^*}{2^{t+1}}\right]=-\frac{2^{t+1}}{(T^*)^2}+\frac{1}{2^{t+1}}<0 \text{ for } T^*<2^{t+1}.$$

So worst possible T^* is where these are equal:

$$-\frac{2^t}{(T^*)^2}+\frac{1}{2^t}=-\frac{2^{t+1}}{(T^*)^2}+\frac{1}{2^{t+1}}.$$

This gives

$$T^*=2^{t+\frac{1}{2}}.$$

The worst isn't that bad

$$
\begin{aligned}
\frac{G(Q)}{G(Q^*)} &\leq \tfrac{1}{2}\frac{2^t}{2^{t+\frac{1}{2}}} + \tfrac{1}{2}\frac{2^{t+\frac{1}{2}}}{2^t} \\
&= \tfrac{1}{2}\frac{1}{2^{\frac{1}{2}}} + \tfrac{1}{2}2^{\frac{1}{2}} \\
&= \tfrac{1}{2}\frac{1}{\sqrt{2}} + \tfrac{1}{2}\sqrt{2} \\
&< 1.061.
\end{aligned}
$$

The worst isn't that bad

Conclusion

Power of two ordering is never more than 6.1% more expensive than if we could order any moment in time.