Sketchy Decisions: Convex Low-Rank Matrix Optimization with Optimal Storage

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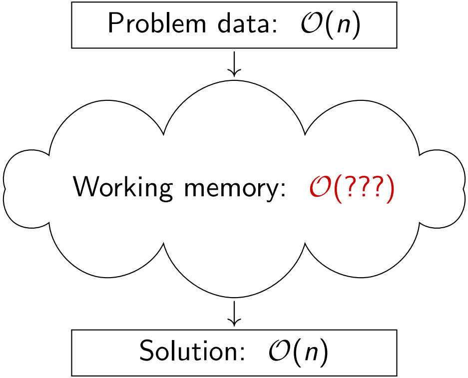
Goal

Can we develop algorithms that provably solve a problem using **storage** bounded by the size of the **problem data** and the size of the **solution**?



Model problem: low rank matrix optimization

consider a convex problem with decision variable $X \in \mathbb{R}^{m \times n}$

compact matrix optimization problem:

$$\begin{array}{ll} \text{minimize} & f(\mathcal{A}X) \\ \text{subject to} & \|X\|_{S_1} \leq \alpha \end{array} \qquad \text{(CMOP)}$$

- $\mathcal{A} : \mathbb{R}^{m \times n} \to \mathbb{R}^d$
- $f : \mathbb{R}^d \to \mathbb{R}$ convex and smooth
- $\|X\|_{S_1}$ is Schatten-1 norm: sum of singular values

assume

- **compact specification**: problem data use $\mathcal{O}(n)$ storage
- **compact solution**: $\operatorname{rank} X_\star = r$ constant

Note: Same ideas work for $X \succeq 0$

Are desiderata achievable?

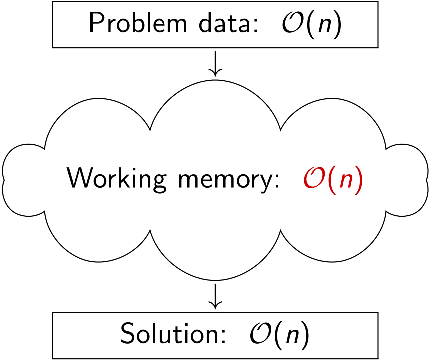
$$\begin{array}{ll} \text{minimize} & f(\mathcal{A}X) \\ \text{subject to} & \|X\|_{S_1} \leq \alpha \end{array}$$

CMOP, using any first order method:

$$\boxed{\text{Problem data: } \mathcal{O}(n)}$$
$$\downarrow$$
$$\boxed{\text{Working memory: } \mathcal{O}(n^2)}$$
$$\downarrow$$
$$\boxed{\text{Solution: } \mathcal{O}(n)}$$

Are desiderata achievable?

CMOP, using **SketchyCGM**:



Application: matrix completion

find X matching M on observed entries

$$\begin{array}{ll} \text{minimize} & \sum_{(i,j)\in\Omega}(X_{ij} - M_{ij})^2 \\ \text{subject to} & \|X\|_{S_1} \leq \alpha \end{array}$$

- $m =$ rows, $n =$ columns of matrix to complete
- $d = |\Omega|$ number of observations
- $\mathcal{A}$ selects observed entries X_{ij}, $(i,j) \in \Omega$
- $f(z) = \|z - \mathcal{A}M\|^2$

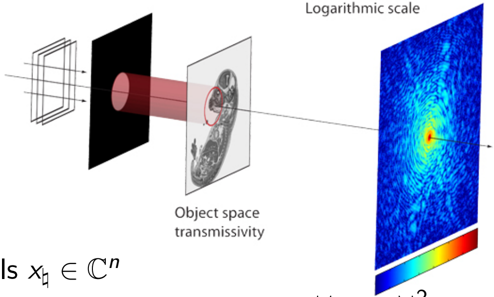
Matrix completion is a CMOP

find X matching M on observed entries

$$\begin{array}{ll} \text{minimize} & \sum_{(i,j)\in\Omega}(X_{ij} - M_{ij})^2 \\ \text{subject to} & \|X\|_{S_1} \leq \alpha \end{array}$$

- compact specification if $d = \mathcal{O}(m + n)$ observations
  e.g., constant $\#$ observations / person
- compact solution if $\text{rank}(X)$ constant
  i.e., constant $\#$ parameters / person
- in practice, usually find rank $\ll 200$ even with m and n in the millions...

Application: Phase retrieval



[1]

- image with n pixels $x_\natural \in \mathbb{C}^n$
- acquire noisy nonlinear measurements $b_i = |\langle a_i, x_\natural \rangle|^2 + \omega_i$
- relax: if $X = x_\natural x_\natural^*$, then

$$|\langle a_i, x_\natural \rangle|^2 = x_\natural a_i^* a_i x_\natural^* = \operatorname{tr}(a_i^* a_i x_\natural^* x_\natural) = \operatorname{tr}(a_i^* a_i X)$$

- recover image by solving

$$\begin{array}{ll} \text{minimize} & f(\mathcal{A}X; b) \\ \text{subject to} & \operatorname{tr} X = \alpha \\ & X \succeq 0. \end{array}$$

[1] image courtesy of Manuel Guizar-Sicairos

Phase retrieval is a CMOP

find X matching observations

$$\begin{array}{ll} \text{minimize} & f(\mathcal{A}X; b) \\ \text{subject to} & \operatorname{tr} X = \alpha \\ & X \succeq 0. \end{array}$$

- compact specification if $d = \mathcal{O}(n)$ observations
  e.g., constant # observations / pixels
- compact solution if $\operatorname{rank}(X)$ constant
  e.g., if correctly recover the rank-1 solution!

Why compact?

why a compact specification?

- data is expensive
- collect constant data per column (=user or sample)
- if solution is compact, compact specification should suffice

why a compact solution?

- the world is simple and structured
- given d observations, there is a solution with rank $\mathcal{O}(\sqrt{d})$
  (Barvinok 1995, Pataki 1998)
- nice latent variable models are of log rank
  (Udell & Townsend 2019)

Optimal Storage

What kind of storage bounds can we hope for?

- Assume black-box implementation of

$$\mathcal{A}(uv^*) \qquad u^*(\mathcal{A}^* z) \qquad (\mathcal{A}^* z)v$$

  where $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$, and $z \in \mathbb{R}^d$
- Need $\Omega(m + n + d)$ storage to apply linear map
- Need $\Theta(r(m+n))$ storage for a rank-r approximate solution

Definition. An algorithm for the model problem has **optimal storage** if its working storage is

$$\Theta(d + r(m+n)).$$

If we write down X, we've already failed.

A brief biased history of matrix optimization (I)

- 1990s: **Interior-point methods**
  - Storage cost $\Theta((m+n)^4)$ for Hessian
- 2000s: **Convex first-order methods (FOM)**
  - (Accelerated) proximal gradient and others
  - Store matrix variable $\Theta(mn)$

(**Interior-point:** Nemirovski & Nesterov 1994; . . . ; **First-order:** Rockafellar 1976; Auslender & Teboulle 2006; . . .)

A brief biased history of matrix optimization (I)

- 2008–Present: **Storage-efficient convex FOM**
  - Conditional gradient method (CGM) and extensions
  - Store matrix in low-rank form $\mathcal{O}(t(m+n))$ after t iterations
  - Requires storage $\Theta(mn)$ for $t \geq \min(m, n)$
  - Variants: prune factorization, or seek rank-reducing steps

- 2003–Present: **Nonconvex methods**
  - Burer–Monteiro factorization idea + various opt algorithms
  - Store low-rank matrix factors $\Theta(r(m+n))$
  - For guaranteed solution, need statistical assumptions or $\mathcal{O}(n^{3/2})$ storage

(**CGM:** Frank & Wolfe 1956; Levitin & Poljak 1967; Hazan 2008; Clarkson 2010; Jaggi 2013; ...; **CGM+pruning:** Rao Shah Wright 2015; Freund Grigas Mazumder 2017; ...; **Nonconvex:** Burer & Monteiro 2003; Keshavan et al. 2009; Jain et al. 2012; Bhojanapalli et al. 2015; Candès et al. 2014; Boumal et al. 2015; Bhojanapalli et al. 2018; Waldspurger & Waters 2018; ...)

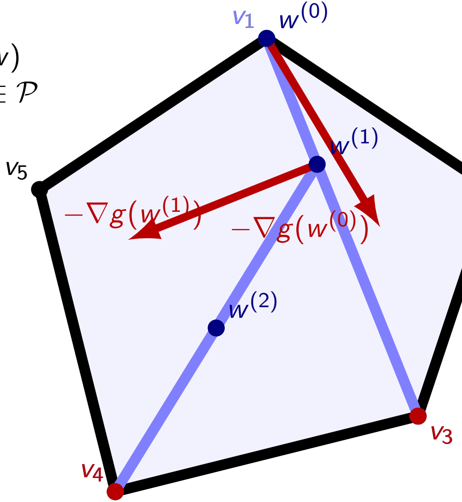
The dilemma

- convex methods: slow memory hogs with guarantees
- nonconvex methods: fast, lightweight, but brittle

goal: low memory and guaranteed convergence

Conditional gradient method (Frank-Wolfe)

$$\begin{array}{ll} \text{minimize} & g(w) \\ \text{subject to} & w \in \mathcal{P} \end{array}$$



Conditional Gradient Method

$$\begin{array}{ll} \text{minimize} & f(\mathcal{A}X) \\ \text{subject to} & \|X\|_{S_1} \le \alpha \end{array}$$

CGM. set $X^0 = 0$. for $t = 0, 1, \ldots$

- compute $G^t = \mathcal{A}^* \nabla f(\mathcal{A}X^t)$
- set search direction

$$H^t = \operatorname*{argmax}_{\|X\|_{S_1} \le \alpha} \langle X, -G^t \rangle$$

- set stepsize $\eta^t = 2/(t+2)$
- update $X^{t+1} = (1 - \eta^t) X^t + \eta^t H^t$

Conditional gradient method (CGM)

features:

- relies on efficient **linear optimization oracle** to compute

$$H^t = \underset{\|X\|_{S_1} \leq \alpha}{\operatorname{argmax}} \langle X, -G^t \rangle$$

- bound on suboptimality follows from subgradient inequality

$$\begin{aligned} f(\mathcal{A}X^t) - f(\mathcal{A}X^\star) &\leq \langle X^t - X^\star, G^t \rangle \\ &\leq \langle X^t - X^\star, \mathcal{A}^* \nabla f(\mathcal{A}X^t) \rangle \\ &\leq \langle \mathcal{A}X^t - \mathcal{A}X^\star, \nabla f(\mathcal{A}X^t) \rangle \\ &\leq \langle \mathcal{A}X^t - \mathcal{A}H^t, \nabla f(\mathcal{A}X^t) \rangle \end{aligned}$$

  to provide stopping condition

- faster variants: linesearch, away steps, ...

Linear optimization oracle for MOP

compute search direction

$$\underset{\|X\|_{S_1} \le \alpha}{\operatorname{argmax}} \langle X, -G \rangle$$

- solution given by maximum singular vector of $-G$:

$$-G = \sum_{i=1}^{n} \sigma_i u_i v_i^* \quad \Longrightarrow \quad X = \alpha u_1 v_1^*$$

- use Lanczos method: only need to apply G and G^*

Conditional gradient descent

Algorithm 1 CGM for the model problem (CMOP)

Input: Problem data for (CMOP); suboptimality ε
Output: Solution $X_\star$

```
function CGM
    X ← 0
    for t ← 0, 1, ... do
        (u, v) ← MaxSingVec(−𝒜*(∇f(𝒜X)))
        H ← −α uv*
        if ⟨𝒜X − 𝒜H, ∇f(𝒜X)⟩ ≤ ε then break for
        η ← 2/(t + 2)
        X ← (1 − η)X + ηH
    return X
```

Two crucial ideas

To solve the problem using optimal storage:

- Use the low-dimensional "dual" variable

$$z_t = \mathcal{A}X_t \in \mathbb{R}^d$$

  to drive the iteration.
- Recover solution from small (randomized) sketch.

Never write down X until it has converged to low rank.

Conditional gradient descent

Algorithm 2 CGM for the model problem (CMOP)

Input: Problem data for (CMOP); suboptimality ε
Output: Solution $X_\star$

function CGM
  $X \leftarrow 0$
  for $t \leftarrow 0, 1, \ldots$ **do**
    $(u, v) \leftarrow \texttt{MaxSingVec}(-\mathcal{A}^*(\nabla f(\mathcal{A}X)))$
    $H \leftarrow -\alpha\, uv^*$
    if $\langle \mathcal{A}X - \mathcal{A}H, \nabla f(\mathcal{A}X)\rangle \le \varepsilon$ **then break for**
    $\eta \leftarrow 2/(t+2)$
    $X \leftarrow (1-\eta)X + \eta H$
  return X

Conditional gradient descent

Introduce "dual variable" $z = \mathcal{A}X \in \mathbb{R}^d$; eliminate X.

Algorithm 3 Dual CGM for the model problem (CMOP)

Input: Problem data for (CMOP); suboptimality ε
Output: Solution $X_\star$

function $\textsc{dual}\mathrm{CGM}$
  $z \leftarrow 0$
  for $t \leftarrow 0, 1, \ldots$ **do**
    $(u, v) \leftarrow \texttt{MaxSingVec}(-\mathcal{A}^*(\nabla f(z)))$
    $h \leftarrow \mathcal{A}(-\alpha uv^*)$
    if $\langle z - h, \nabla f(z)\rangle \leq \varepsilon$ **then break for**
    $\eta \leftarrow 2/(t+2)$
    $z \leftarrow (1-\eta)z + \eta h$

we've solved the problem. . . but where's the solution?

Two crucial ideas

1. Use the low-dimensional "dual" variable

$$z_t = \mathcal{A}X_t \in \mathbb{R}^d$$

   to drive the iteration.
2. Recover solution from small (randomized) sketch.

How to catch a low rank matrix

if $\hat{X}$ has the same rank as $X^\star$,
and $\hat{X}$ acts like $X^\star$ (on its range and co-range),
then $\hat{X}$ is $X^\star$

use single-pass randomized sketch (Tropp Yurtsever U Cevher 2017)

- see a series of additive updates
- remember how the matrix acts on random subspace
- reconstruct a low rank matrix that acts like $X^\star$
- storage cost for sketch and arithmetic cost of update are $\mathcal{O}(r(m+n))$; reconstruction is $\mathcal{O}(r^2(m+n))$

Single-pass randomized sketch

- Draw and fix two independent standard normal matrices
$$\Omega \in \mathbb{R}^{n\times k} \quad \text{and} \quad \Psi \in \mathbb{R}^{\ell\times m}$$
with $k = 2r + 1$, $\ell = 4r + 2$.
- The sketch consists of two matrices that capture the range and co-range of X:
$$Y = X\Omega \in \mathbb{R}^{n\times k} \quad \text{and} \quad W = \Psi X \in \mathbb{R}^{\ell\times m}$$
- Rank-1 updates to X can be performed on sketch:
$$X' = \beta_1 X + \beta_2 uv^*$$
$$\Downarrow$$
$$Y' = \beta_1 Y + \beta_2 uv^*\Omega \quad \text{and} \quad W' = \beta_1 W + \beta_2 \Psi uv^*$$
- Both the storage cost for the sketch and the arithmetic cost of an update are $\mathcal{O}(r(m + n))$.

Recovery from sketch

To recover rank-r approximation $\hat{X}$ from the sketch, compute

1. $Y = QR$ (tall-skinny QR)
2. $B = (\Psi Q)^\dagger W$ (small QR + backsub)
3. $\hat{X} = Q[B]_r$ (tall-skinny SVD)

Theorem (Reconstruction (Tropp Yurtsever U Cevher, 2016))

Fix a target rank r. Let X be a matrix, and let (Y, W) be a sketch of X. The reconstruction procedure above yields a rank-r matrix $\hat{X}$ with

$$\mathbb{E}\,\|X - \hat{X}\|_{\mathrm{F}} \leq 2\,\|X - [X]_r\|_{\mathrm{F}}\,.$$

Similar bounds hold with high probability.

Previous work (Clarkson Woodruff 2009) algebraically but not numerically equivalent.

Recovery from sketch: intuition

let

$$Y = X\Omega \in \mathbb{R}^{n \times k} \quad \text{and} \quad W = \Psi X \in \mathbb{R}^{\ell \times m}$$

- if Q is an orthonormal basis for $\mathcal{R}(X)$, then

$$X = QQ^*X$$

- if $QR = X\Omega$, then Q is (approximately) a basis for $\mathcal{R}(X)$
- and if $W = \Psi X$, we can estimate

$$\begin{aligned} W &= \Psi X \\ &\approx \Psi QQ^*X \\ (\Psi Q)^\dagger W &\approx Q^*X \end{aligned}$$

- hence we may reconstruct X as

$$X \approx QQ^*X \approx Q(\Psi Q)^\dagger W$$

SketchyCGM

Algorithm 4 SketchyCGM for the model problem (CMOP)

Input: Problem data; suboptimality ε; target rank r
Output: Rank-r approximate solution $\hat{X} = U\Sigma V^*$

function $\textsc{SketchyCGM}$
    $\textsc{Sketch.Init}(m, n, r)$
    $z \leftarrow 0$
    for $t \leftarrow 0, 1, \dots$ **do**
        $(u, v) \leftarrow \texttt{MaxSingVec}(-\mathcal{A}^*(\nabla f(z)))$
        $h \leftarrow \mathcal{A}(-\alpha u v^*)$
        if $\langle z - h, \nabla f(z) \rangle \le \varepsilon$ **then break for**
        $\eta \leftarrow 2/(t+2)$
        $z \leftarrow (1-\eta) z + \eta h$
        $\textsc{Sketch.CGMUpdate}(-\alpha u, v, \eta)$
    $(U, \Sigma, V) \leftarrow \textsc{Sketch.Reconstruct}(\,)$
    return (U, Σ, V)

Guarantees

Suppose

- $X_{\mathrm{cgm}}^{(t)}$ is tth CGM iterate
- $\lfloor X_{\mathrm{cgm}}^{(t)} \rfloor_r$ is best rank r approximation to CGM solution
- $\hat{X}^{(t)}$ is SketchyCGM reconstruction after t iterations

Theorem (Convergence to CGM solution)

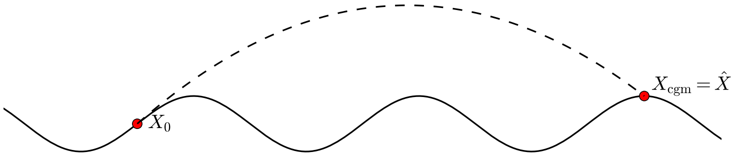
After t iterations, the SketchyCGM reconstruction satisfies

$$\mathbb{E}\,\|\hat{X}^{(t)} - X_{\mathrm{cgm}}^{(t)}\|_{\mathrm{F}} \leq 2\,\|\lfloor X_{\mathrm{cgm}}^{(t)} \rfloor_r - X_{\mathrm{cgm}}^{(t)}\|_{\mathrm{F}}\,.$$

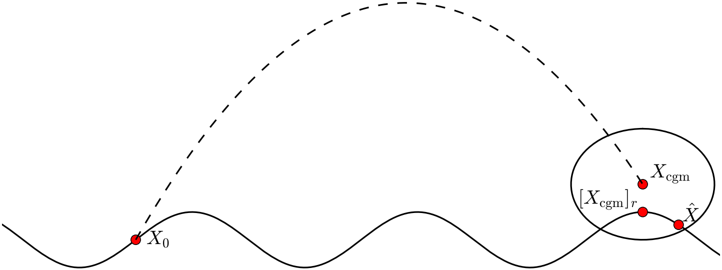
If in addition $X^\star = \lim_{t\to\infty} X_{\mathrm{cgm}}^{(t)}$ has rank r, then RHS $\to 0$!

(Tropp Yurtsever U Cevher, 2016)

Convergence when $\text{rank}(X_{\mathbf{cgm}}) \leq r$



Convergence when $\mathrm{rank}(X_{\mathbf{cgm}}) > r$



Guarantees (II)

Theorem (Convergence rate)

Fix $\kappa > 0$ and $\nu \geq 1$. Suppose the (unique) solution $X_\star$ of (CMOP) *has* $\text{rank}(X_\star) \leq r$ *and*

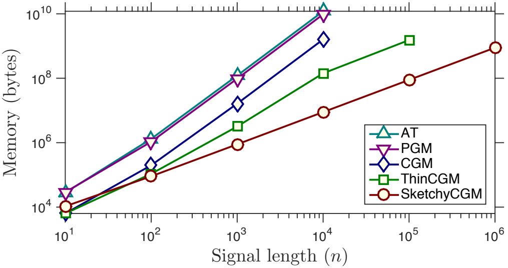
$$f(\mathcal{A}X) - f(\mathcal{A}X_\star) \geq \kappa \, \|X - X_\star\|_{\mathrm{F}}^{\nu} \quad \textit{for all} \quad \|X\|_{S_1} \leq \alpha. \quad (1)$$

Then we have the error bound

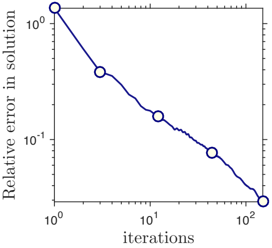
$$\mathbb{E}\, \|\hat{X}_t - X_\star\|_{\mathrm{F}} \leq 6 \left(\frac{2\kappa^{-1} C}{t+2} \right)^{1/\nu} \quad \textit{for } t = 0, 1, 2, \ldots$$

where C is the curvature constant (Eqn. (3), Jaggi 2013) of the problem (CMOP).

SketchyCGM is scalable



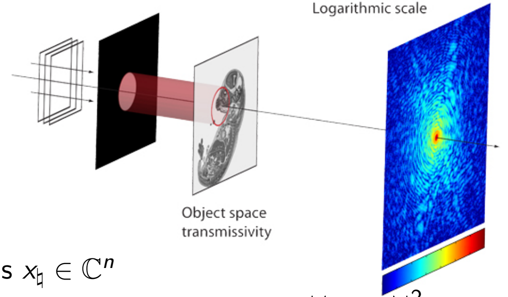
(A) Memory usage for five algorithms



(B) Convergence for $n = 8 \cdot 10^6$.

PGM	$=$	proximal gradient (via TFOCS (Becker Candès Grant, 2011))
AT	$=$	accelerated PGM (Auslander Teboulle, 2006) (via TFOCS),
CGM	$=$	conditional gradient method (Jaggi, 2013)
ThinCGM	$=$	CGM with thin SVD updates (Yurtsever Hsieh Cevher, 2015)
SketchyCGM	$=$	ours, using $r = 1$

Application: Phase retrieval



- image with n pixels $x_\natural \in \mathbb{C}^n$
- acquire noisy nonlinear measurements $b_i = |\langle a_i, x_\natural \rangle|^2 + \omega_i$
- relax: if $X = x_\natural x_\natural^*$, then

$$|\langle a_i, x_\natural \rangle|^2 = x_\natural a_i^* a_i x_\natural^* = \mathrm{tr}(a_i^* a_i x_\natural^* x_\natural) = \mathrm{tr}(a_i^* a_i X)$$

- recover image by solving

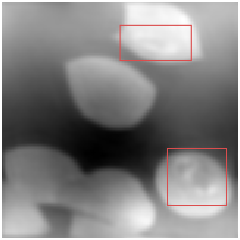
$$\begin{array}{ll} \text{minimize} & f(\mathcal{A}X; b) \\ \text{subject to} & \mathrm{tr}\, X \le \alpha \\ & X \succeq 0. \end{array}$$

compact if $d = \mathcal{O}(n)$ observations and $\mathrm{rank}(X^\star)$ constant

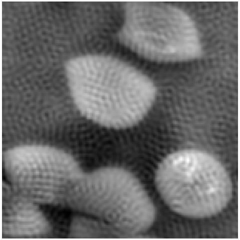
SketchyCGM is reliable

Fourier ptychography:

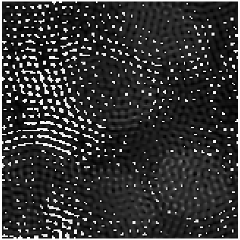
- imaging blood cells with $\mathcal{A}$ = subsampled FFT
- $n = 25,600$, $d = 185,600$
- $\text{rank}(X_\star) \approx 5$ (empirically)



(A) SketchyCGM



(B) Burer–Monteiro



(C) Wirtinger Flow

- brightness indicates phase of pixel (thickness of sample)
- red boxes mark malaria parasites in blood cells

Conclusion

SketchyCGM offers a proof-of-concept **convex method** with **optimal storage** for low rank matrix optimization using two new ideas:

- Drive the algorithm using a smaller (dual) variable.
- Sketch and recover the decision variable.

References:

- J. A. Tropp, A. Yurtsever, M. Udell, and V. Cevher. Randomized single-view algorithms for low-rank matrix reconstruction. SIMAX 2017.
- A. Yurtsever, M. Udell, J. A. Tropp, and V. Cevher. Sketchy Decisions: Convex Optimization with Optimal Storage. AISTATS 2017.